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APPLICATION OF THE SAMUELSON EQUATION TO THE EVANS MODEL

ЗАСТОСУВАННЯ РІВНЯННЯ САМУЕЛСОНА В МОДЕЛІ ЕВАНСА

Mathematical modeling of economic processes is an actual direction of research, because the well-being of citizens and the country as a whole depends on it. In the case of the market, the prices of most goods and services are not planned centrally, are not directly regulated by the state, but are freely set and changed by the market itself. The main factors that control the movement of prices in the market are the demand and supply of goods. In the economy, the most important are dynamic models, the parameters of which change over time. The Evans model (Walras-Evans-Samuelson model) is currently one of the basic concepts explaining the dynamic establishment of the equilibrium price in the market of one product under the influence of supply and demand. This is due to the fact that knowing the dynamics of the economic parameter we are interested in, we can try to build a forecast of its further evolution. The article examines the market of one product. For convenience, we will assume that the functions of the dependence of demand and supply on the price are given by linear relationships. The construction of the Evans model is that the change in price is directly proportional to the excess of demand over supply and the duration of this excess. The Samuelson differential equation with an initial condition, that is, the Cauchy problem, is obtained. Samuelson's equation has a stationary (equilibrium) point, which is a positive price at which supply and demand will be equal. The analysis of the obtained solution of the problem shows that over a long enough time (relatively speaking, at $t \to \infty$) the price asymptotically approaches the equilibrium value. If we are not interested in the temporal dependence, but only in the equilibrium price, then it can be found from the differential equation immediately by setting the condition $\frac{dp}{dt} = 0$, this is the so-called limit stationary mode. The solution of the Cauchy problem is found by the method of variation of the constant. The parameterization of the model using the fractional differentiation operator in the sense of Gerasimov-Caputo is considered. The resulting solution was analyzed depending on the parameter $0 < \alpha \leq 1$. For this, the asymptotic representation of the function was used for large values of the argument. Conclusions are made regarding price dynamics in time relative to the equilibrium price when demand and supply are equal.

Key words: *dynamical system, demand function, offer function, Evans model, Samuelson's equation.*

Математичне моделювання економічних процесів є актуальним напрямом дослідження, тому що від цього залежить добробут громадян і країни в цілому. У разі ринку, ціни більшості товарів та послуг не плануються централізовано, не регулюються безпосередньо державою, а вільно встановлюються і змінюються самим ринком. Основними факторами, що керують рухом цін на ринку, є попит та пропозиція товарів. В економіці найбільш важливими є динамічні моделі, параметри яких змінюються в часі. Модель Еванса (модель Вальраса-Еванса-Самуельсона) нині є однією з базових концепцій, що пояснюють динамічне встановлення рівноважної ціни на ринку одного товару під впливом попиту та пропозиції. Це зумовлено тим, що знаючи динаміку економічного параметра, що цікавить нас, можна спробувати побудувати прогноз його подальшої еволюції. У статті розглядається ринок одного товару. Для зручності вважатимемо, що функції залежності попиту *та пропозиції від ціни задані лінійними співвідношеннями. Побудова моделі Еванса полягає в тому, що зміна ціни прямо пропорційна до перевищення попиту над пропозицією і тривалості цього перевищення. Отримано диференціальне рівняння Самуельсона з початковою умовою, тобто задача Коші. Рівняння Самуельсона має стаціонарну (рівноважну) точку, що є додатньою ціною, при якій попит і пропозиції будуть рівні. Аналіз отриманого рішення задачі показує, що за досить тривалого часу (умовно кажучи, при* $t \to \infty$*) ціна асимптотично наближається до рівноважного значення. Якщо нас цікавить не тимчасова залежність, а лише рівноважна ціна, то її можна знайти з диференціального рівняння відразу, задавши умову* $\frac{dp}{dt} = 0$*, це так званий граничний стаціонарний режим. Розв'язання задачі Коші знаходиться методом варіації постійної. Розглядається параметризація моделі за допомогою оператора дробового диференціювання в сенсі Герасимова – Капуто. Отримане рішення проаналізовано в залежності від параметра* $0 < \alpha \leq 1$*. Для цього використано асимптотичне представлення функції при великих значеннях аргументу. Зроблено висновки щодо динаміки ціни у часі відносно рівноважної ціни, коли попит та пропозиції рівні.*

Ключові слова: *динамічна система, функція попиту, функція пропозиції, модель Еванса, рівняння Самуельсона.*

Formulation of the problem. Dynamic models are described mainly by linear ordinary differential equations with initial conditions, which are easily solved by known methods. As a rule, the solution to such equations is an exponential with negative or positive exponents, depending on the economic meaning. However, mathematical models appear that may contain derivatives of fractional orders in their equations. We will not dwell in detail on the question of the economic meaning of such differential operators, but let's just consider the features of solving such equations using the Evans model as an example.

Analysis of recent research and publications. By modeling a certain process we mean its display in the language of the corresponding science. A mathematical model is a set of symbolic mathematical objects and relationships between them. In economics, these are economic-mathematical models. Modeling is the main method of economic equilibrium research. The need to use certain models stems from the essence of equilibrium states in the economy. The hypothesis about the existence of general economic equilibrium was originally stated in the work of L. Walras [1]. Pure political economy was conceived by Walras as a theory of price determination under a hypothetical regime of free competition. This theory is mathematical, which means that although it can be stated in ordinary language, its proof must be mathematical. It is entirely based on the theory of exchange, which, in turn, is entirely expressed – in a state of market equilibrium – in a double fact (the central idea of the theory of general equilibrium): firstly, the fact that each participant in the exchange receives maximum utility, and secondly, in the fact equality of supply and demand volumes for each product for all participants. Only mathematics can give us the condition for maximum utility. These two facts underlie the equilibrium state in markets, according to L. Walras. If the first fact defines equilibrium as the state of each individual economic entity, then the second fact fixes equilibrium as the state of the system of interacting entities as a whole. As new research appears, theoretical explanations of general economic equilibrium inevitably become more complex, incorporating approaches from related disciplines: game theory, algebra and linear programming. The theory of general economic equilibrium is becoming the leading approach for modeling socio-economic processes [2–3]. General economic equilibrium models are actively used as an applied tool when analyzing the efficiency of the economy, as well as the influence of various kinds of exogenous influences and endogenous changes on the economy at various levels [3–4]. Despite significant differences in scientific approaches to the study of economic equilibrium, in most scientific sources its essence is reduced to the balance and proportionality of economic processes (economic relations) within a certain economic system. Thus, in fact, we are investigating different states of some system consisting of subjects, relations between them and objects in relation to which these relations are realized [5–7]. Mathematical models can be studied by various methods: analytical, which obtain the general form of dependencies between the research characteristics or their properties, numerical, etc. Models of general equilibrium, used to obtain quantitative estimates, are divided into two types: static and dynamic stochastic. In particular, static models characterize the specific state of the object at a given moment in time, stochastic models reflect the influence of random factors [8]. Dynamic mathematical modeling is a description of time-varying processes using differential, integral, integro-differential, stochastic differential equations, in particular. Sometimes such functional relations themselves reflect the law obtained during the study of the process by the means of the corresponding field in natural science or social sciences [9]. Although more often final mathematical models are obtained in the process of applications of mathematics itself, logic, and therefore there is a need for their verification in practice – testing. In modeling, three main parts are distinguished: empirical, theoretical, mathematical, which constructs mathematical models to explain and verify the theoretical part, process experimental data, and plan further research. At the initial stage, the models are simplified in order to choose the appropriate mathematical apparatus for research, give the solution algorithm and test it. It turned out that if we replace some derivatives of whole orders with certain integral relations, in particular, non-local derivatives of fractional orders, then the gradualness of the process will be taken into account. In recent years, models that are described using equations with fractional derivatives have been actively studied [9–10].

Formulation of the purpose of the article. Analyze the possibility of solving differential equations containing derivatives of fractional orders in the Evans dynamic model.

Presenting main material. *Mathematical formulation of the problem*. Consider the market for one product. Let us introduce into consideration the following defining functions $D(t)$, $S(t)$, and $p(t)$, which in economics are known as demand, supply and price [11]. We will also assume that supply and demand linearly depend on price according to the equations:

$$D(t) = a0 - bp(t),$$
$$S(t) = a1 + \beta p(t), \tag{1}$$
$$\alpha, \beta, a, b > 0$$

It should be noted that if $p(t) = 0$, then $a > \alpha$, i.e. demand prevails over supply at zero price. In the Evans model, the determining factor is the change in price depending on the relationship between supply and demand [12]. The change in price over time t must be proportional to the excess of demand over supply, i.e. the following equation holds:

$$\frac{dp}{dt} = \lambda\left(D(t) - S(t)\right), \tag{2}$$

here $\lambda > 0$ is the proportionality coefficient. The equation in our case can be written as follows:

$$\frac{dp}{dt} = -\lambda\left((b+\beta)\, p(t) - a0 + a1\right), \tag{3}$$

Equation (3) is called Samuelson's equation. To determine the integration constant in (3), it is necessary to set the initial condition

$$p(0) = p0. \tag{4}$$

Relations (3) and (4) define the Cauchy problem. From equation (3) we can easily determine the equilibrium price ($D = S$), assuming $\frac{dp}{dt} = 0$, we arrive at the following result:

$$\bar{p} = \frac{a0 - a1}{b + \beta} > 0. \quad (5)$$

The equilibrium price (5) has the property that for $p0 > \bar{p}$ the price p increases as it tends to the equilibrium price, and for $p0 < \bar{p}$ the price decreases accordingly. The solution to the Cauchy problem (3), (4) can be found by the method of varying the constant

$$p(t) = p0e^{-\lambda(\beta+b)t} + \frac{a0 - a1}{b + \beta}\left(1 - e^{-\lambda(\beta+b)t}\right). \quad (6)$$

It is known that $\lim_{t\to\infty} p(t) = \bar{p}$.

Let us consider the parameterization of model (3) and (4). Consider the following Cauchy problem:

$$\partial_{0t}^{\alpha} p(\tau) = -\lambda\left((b+\beta)p(t) - a0 + a1\right),$$

$$p(0) = p0,$$

$$0 < \alpha < 1. \quad (7)$$

Here $\partial_{0t}^{\alpha} p(\tau) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{p'(\tau)d\tau}{(t-\tau)^{\alpha}}$ is the fractional differentiation operator Gerasimova – Caputo [9].

The choice of the differential operator is due to the following reasons: 1) the possibility of applying the initial local condition (4); 2) the derivative of order α from a constant is equal to zero. Let us write the Cauchy problem (7) in the form:

$$\partial_{0t}^{\alpha} p(\tau) = -\lambda(b+\beta)p(t) + \lambda(a0 - a1),$$

$$p(0) = p0,$$

$$0 < \alpha < 1. \quad (8)$$

The solution of the problem.

The solution to problem (8) can be written as follows:

$$p(t) = p0E_{\alpha,1}\left(-\lambda(b+\beta)t^{\alpha}\right) + \lambda(a0 - a1)\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(b+\beta)(t-\tau)^{\alpha}\right)d\tau;$$

$$p0E_{\alpha,1}\left(-\lambda(b+\beta)t^{\alpha}\right) + \lambda(a0 - a1)\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(b+\beta)(t-\tau)^{\alpha}\right)d\tau =$$

$$= p0E_{\alpha,1}\left(-\lambda(b+\beta)t^{\alpha}\right) + \lambda t^{\alpha}(a0 - a1)E_{\alpha,\alpha+1}\left(-\lambda(b+\beta)t^{\alpha}\right) =$$

$$= p0E_{\alpha,1}\left(-\lambda(b+\beta)t^{\alpha}\right) + \frac{a0 - a1}{b+\beta}\left(1 - E_{\alpha,1}\left(-\lambda(b+\beta)t^{\alpha}\right)\right);$$

where $\bar{p} = \frac{a0 - a1}{b+\beta}$,

$$p0E_{\alpha,1}\left(-\lambda(b+\beta)t^{\alpha}\right) + \bar{p}\left(1 - E_{\alpha,1}\left(-\lambda(b+\beta)t^{\alpha}\right)\right) =$$

$$= (p0 - \bar{p})E_{\alpha,1}\left(-\lambda(b+\beta)t^{\alpha}\right) + \bar{p},$$

$$p(t) = (p0 - \bar{p})E_{\alpha,1}\left(-\lambda(b+\beta)t^{\alpha}\right) + \bar{p}. \quad (9)$$

Here $E_{\alpha,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + \alpha k)}$ – is a Mittag-Leffler type function,

$\Gamma(z)$ – is a gamma Euler function. In solution (9) the property of a Mittag-Leffler type function was used $E_{\alpha,\mu}(z) = zE_{\alpha,\alpha} + \mu(z) + \frac{1}{\Gamma(\mu)}$ [13].

It should be noted that with the value of the parameter $\alpha = 1$, the solution, up to a factor, will transform into solution (6). Let us show that solution (9) tends to (5) as $t \to \infty$. To do this, we use the asymptotic representation of the function for large values of the argument:

$$E_{\alpha,1}\left(-\lambda\left(b+\beta\right)t^{\alpha}\right) = \frac{1}{\alpha}e^{\left[-\lambda(b+\beta)\right]^{1/\alpha_t}},$$

$$|z| = \left|\lambda\left(b+\beta\right)t^{\alpha}\right| \to \infty. \quad (10)$$

Then, substituting (10) into (9) and at $t \to \infty$ we arrive at the limit $\lim_{t\to\infty} p(t) = \bar{p}$.

Conclusions. Solution (9) differs from solution (6) in the arbitrariness of the choice of parameter $0 < \alpha \leq 1$.

In equation (7), an integral with a power kernel means a rather tricky averaging of the price of a product over time or non-locality over time. This results in a slowdown in price dynamics over time relative to the equilibrium price when supply and demand are equal. Such a slowdown may be caused by some external factors or a feature of the monopoly.

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