МАТЕМАТИЧНІ МЕТОДИ, МОДЕЛІ ТА ІНФОРМАЦІЙНІ ТЕХНОЛОГІЇ В ЕКОНОМІЦІ

UDC 519.86
DOI: https://doi.org/10.32782/2708-0366/2022.13.25

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MATHEMATICAL MODEL OF THE MARKET OF ONE PRODUCT WITH OPTIMAL DELIVERY TO THE MARKET UNDER CONDITIONS OF DELAY

МАТЕМАТИЧНА МОДЕЛЬ РИНКУ ОДНОГО ТОВАРУ З ОПТИМАЛЬНИМ ПОСТАЧАННЯМ ТОВАРУ НА РИНОК В УМОВАХ ЗАПІЗНЕННЯ

The market of one commodity functioning in discrete time t is considered. It is assumed that the ordered goods enter the market with a delay of τ units of time. In this formulation of the description of market dynamics, a supply line is not required. The problem of mathematical description and simulation modeling of the inertial market of one product under optimal control of the supply of goods to the market under conditions of delayed supplies is considered. The mathematical description of the market is represented by a restrictive (subject to inequality-type constraints) dynamic model with retarded control. It is shown that the optimal strategy for supplying goods to the market in terms of the maximum profit of the seller is determined by the mathematically formulated strict conditions for the state of the market (commodity shortage, overstocking of the market, dynamic equilibrium of the market).

Key words: *dynamical system, inequality type constraints, delayed control, optimization, simulation modeling.*

Розглядається ринок одного товару, що функціонує у дискретному часі t. Передбачається, що замовлений товар надходить на ринок із запізненням на τ одиниць часу. У цій постановці опису динаміки ринку не потрібна лінія пропозиції. Це вже використовувалося в моделях ринку із субоптимальними стратегіями постачання товару на ринок. Розв'язана задача математичного опису та імітаційного моделювання інерційного ринку одного товару при оптимальному управлінні поставкою товару на ринок за умов запізнення поставок. Математичний опис ринку є рестриктивною (що підкоряється обмеженням типу нерівностей) динамічною моделлю із запізнюючим управлінням. Знайдено умовно-оптимальну ціну товару в області товарного дефіциту, в області затоварювання ринку та в області балансу попиту та пропозиції (тобто у сфері динамічної рівноваги). Знайдено оптимальну ціну товару та оптимальний рівень поставки товару на ринок, які забезпечують максимальний прибуток продавця. Вирішення цього завдання проведено по зонах (по зоні 1 – зоні дефіциту товару, зоні 2 – зоні затоварювання ринку, зоні 3 – зоні балансу попиту та пропозиції, тобто динамічної ринкової рівноваги). Очевидно, глобальний максимум прибутку може бути отриманий тільки в тому випадку, якщо обсяг залишку товару, не проданого на попередньому інтервалі часу, не перевищує величини оптимального обсягу постачання товару на ринок. Інакше прибуток продавця буде меншим за максимально *можливий. Причому, якщо при цьому обсяг залишків товару залишатиметься в зоні 3, то ринок залишатиметься у стані динамічної рівноваги (попит на товар залишатиметься рівним попиту, яким буде виступати залишок товару). І тільки коли пропозиція товару перейде до зони 2, почнеться затоварення ринку. Запропонована рестриктивна динамічна математична модель ринку одного товару якісно правильно визначає поведінку ринку в умовах запізнення постачання товару на ринок. Оптимальна стратегія поставки товару вимагає передбачення ціни товару та купівельного попиту вперед на час запізнення, що може бути зроблено за допомогою імітаційного моделювання поведінки ринку.*

Ключові слова: *динамічна система, обмеження типу нерівностей, запізнювальне управління, оптимізація, імітаційне моделювання.*

Formulation of the problem. In the general case, the market mechanism is influenced by many factors: the tastes and preferences of consumers, the interests of sellers, competition between sellers, the level of market monopolization, government legislation, seasonal changes. Some of them are random. All these factors cannot be taken into account. Consider the market mechanism from the point of view of a seller selling one product. The seller decides what price to set, how much product to offer to consumers. As a result, having sold the goods at a favorable price, he can make a big profit, or vice versa, remain with unsold goods, incur losses. It is obvious that three cases are possible: a shortage of goods, an excess of goods, and an equilibrium state. A difficult task arises for the seller: on the one hand, he must not miss his opportunities to make a profit, on the other hand, he must not waste resources in vain, not waste the available funds. The study of economic phenomena is an interesting and difficult task. The peculiarity is that the study of these processes on real objects can be difficult or even impossible and leads to costs, so you need to look for ways to avoid these difficulties. One way to solve this problem is to estimate unknown quantities through simulation.

Analysis of recent research and publications. There are many models for establishing an equilibrium price in the market for one product. The most famous equilibrium models are those of L. Walras and A. Marshall, the "spider" model with discrete time and the Evans model with continuous time [1–2]. The mathematical justification of the Walrasian hypothesis was substantiated in the works of Arrow-Debré, Mackenzie, Gale, Nikaido. In the future, work was carried out to improve the models and their generalization. These studies are considered quite fully in the monographs of Morishima, Nikaido, Lancaster, and modern authors [3]. Most of these works analyzed the balance of aggregate supply and demand [4–5]. The market model should reflect not only the balance between supply and demand, but also the purposefulness of each market participant, taking into account their overall relationship. Such a mathematical model, which, along with the balance sheet, can reflect the purposefulness of each market participant, is a vector (multi-criteria) problem of mathematical programming [6]. To solve this problem, methods for solving a vector problem based on the normalization of criteria and the principle of a guaranteed result have been developed [7–9].

Formulation of the goals of the article. Study of the impact on the dynamics of the price of goods of random fluctuations in demand, the position of the demand line, the purchase price of goods, the strategy of ordering goods. Statistical evaluation of the seller's profit at a fixed price of goods and a strategy for ordering goods. Determination of the optimal price and order strategy, taking into account the delay.

Presenting main material. *1. Statement of the problem.* Let us consider a market for one product operating in discrete time $t \in N = \{0,1,2,\ldots\}$. Let $P(t)$ – the price of a commodity at a point in time t, $Q^o(t)$ – the balance of unsold goods at this point in time, $Q^Z(t-\tau)$ – volume of goods, which is ordered at time $(t-\tau)$ for delivery to the market at time t (delivery strategy). It is assumed that the ordered product enters the market with a delay of τ units of time. Demand for a good at a price $P(t)$ will be denoted by $Q^D(t)$. Let at the moment t of discrete time the demand for goods has the form of the simplest linear dependence:

$$Q^D(t) = Q_m - aP(t), \quad (1)$$

Where $Q_m > 0$, $a > 0$, given constants.

Let $Q(t)$ be the volume of goods offered for sale at time t. Let us represent $Q(t)$ as the sum of the balance of goods in volume $Q^o(t)$ from sales in the previous discrete time interval (and entered the market at time t) and goods in volume $Q^Z(t-\tau)$ ordered by the seller at time $(t-\tau)$ (taking into account the delivery delay) and put it on the market by time *t*: $Q(t) = Q^O(t) + Q^Z(t-\tau)$.

Let us denote the volume of sales on the interval *t* of discrete time as $Q^S(t)$.

Obviously,

$$Q^S(t) = \min\left(Q^D(t), Q^O(t) + Q^Z(t-\tau)\right). \quad (2)$$

Balances of goods satisfy the recurrent relation

$$Q^O(t+1) = Q^O(t) + Q^Z(t-\tau) - Q^S(t). \quad (3)$$

Let $J(t)$ be the seller's profit on the tth interval of discrete time, equal to the difference between the proceeds from the sale of goods and the cost of its purchase and storage. If P_1 is the purchase price of the goods (on the wholesale market or from the manufacturer), P_2 is the storage price of a unit of goods not sold in the previous discrete time interval, the seller's profit at time *t* will be

$$J(t) = Q^S(t)P(t) - Q^Z(t-\tau)P_1 - Q^O(t)P_2 - \frac{R}{2}\left(P(t) - P(t-1)\right)^2. \quad (4)$$

The last term (with a coefficient $R > 0$) expresses "penalties" for changes in commodity prices and determines the inertia of the market – for a sharp rise in price may be followed by legislative sanctions, for a sharp drop in price – "sanctions" of competitors the amount equivalent to this penalty function.

The question arises, what value will take the price of goods $P(t)$ at time *t*, if in the previous $(t-1)$-th step it was equal to $P(t-1)$, and what value $Q^Z(t-\tau)$ of additional supply of goods on the market must make the seller so that the profit of the seller at a given line of demand on the t-th interval of discrete time was the maximum:

$$J(t) \Rightarrow \sup_{P(t), Q^Z(t-\tau)}. \quad (5)$$

Note that in such a statement to describe the dynamics of the market does not require knowledge of the supply line (unlike the classical model of Walras – Marshall [1]).

When solving the optimization problem, the values of product price $P(t)$, sales volume $Q^S(t)$, unsold product balances $Q^O(t)$, and seller's profit $J(t)$ are automatically obtained for each current moment of discrete time *t*. At the same time, of course, restrictions on the value of the possible price of goods $P(t)$ must be met:

$$P_1 < P_{\min} < P(t) < P_{\max} = Q_m / a \quad (6)$$

and the number of additional orders for good $Q^Z(t-\tau) \geq 0$.

2. Conditionally optimal price of goods

Let the volume of supply of goods on the market at time t be $Q(t)$. Find the optimal (which provides the maximum profit of the seller (4)) price $P(t)$ of the product at a fixed value of $Q(t)$: $J(t) \Rightarrow \max_{P(t)/Q(t)}$.

In solving this problem, given its restrictive nature due to the relationship (2), obviously, it is necessary to identify areas corresponding to the shortage of goods on the market (area 1, in which $Q(t) < Q^D(t)$), market overstocking (area 2, in which $Q(t) > Q^D(t)$) and supply and demand balance (area 3, dynamic equilibrium area, in which $Q(t) = Q^D(t)$).

1) In the area of trade deficit $Q(t) < Q^D(t)$ and according to relation (2) we have $Q^S(t) = Q(t)$, so

$$J(t)=Q(t)P(t)-Q(t)P_1+Q^O(t)(P_1-P_2)-\frac{R}{2}(P(t)-P(t-1))^2 \Rightarrow \sup_{P(t)|Q(t)}. \quad (7)$$

This is a quadratic function of the variable $P(t)$, convex upwards. Its maximum is reached at the point

$$P(t)=P(t-1)+\frac{Q(t)}{R}=P^{(1)}(t). \quad (8)$$

As we can see, $P^{(1)}(t)$ increases with increasing $Q(t)$ according to the linear law. Expression (8) is valid not for any $Q(t)$, but only for $Q(t)$, which satisfies the condition $Q(t)<Q^D(t)$ of belonging to domain 1. This condition, taking into account (1) and (8) has the form

$$Q(t)<\frac{R(Q_m-aP(t-1))}{a+R}=Q^{(1)}(t). \quad (9)$$

Thus, in the region 1 $P(t)$ increases linearly with increasing $Q(t)$ from the value $P^{(1)}(t)\Big|_{Q(t)=0}=P(t-1)$ to the value of $P^{(1)}(t)\Big|_{Q(t)=Q^{(1)}(t)}=\frac{Q_m+RP(t-1)}{a+R}=P^{(1)}_{\max}(t)$, moreover, the condition that $Q(t)$ belongs to domain 1 is expressed by inequality (9).

2) In the field of overstocking of the market $Q(t)>Q^D(t)$ and according to expression (2) we have $Q^S(t)=Q^D(t)$, so that taking into account (1)

$$J(t)=(Q_m-aP(t))P(t)-Q(t)P_1+Q^O(t)(P_1-P_2)-R/2(P(t)-P(t-1))^2 \Rightarrow \sup_{P(t)|Q(t)}. \quad (10)$$

This is a quadratic convex upward function of the variable $P(t)$. Its maximum is reached at the point

$$P(t)=\frac{Q_m+RP(t-1)}{2a+R}=P^{(2)}(t). \quad (11)$$

As we can see, $P^{(2)}(t)$ does not depend on $Q(t)$ (remains constant for any $Q(t)$ in this region). Expression (11) is valid only if $Q(t)>Q^D(t)$, ie

$$Q(t)>\frac{R(Q_m-aP(t-1))+aQ_m}{2a+R}=Q^{(2)}(t). \quad (12)$$

The last inequality determines the condition for $Q(t)$ to belong to domain 2. Moreover, $Q^{(2)}(t)>Q^{(1)}(t)$. In fact, using expressions (3) and (8), we obtain $Q^{(2)}(t)-Q^{(1)}(t)=\frac{a^2(Q_m+RP(t-1))}{(a+R)(2a+R)}>0$, which had to be proved.

3) In the area of supply and demand balance (ie in the area of dynamic market equilibrium) $Q(t)=Q^D(t)$ and according to expression (2) we have, as in area 2, sales equal to demand, ie $Q^S(t)=Q^D(t)$, and profit $J(t)$ as (10). However, $Q(t)=Q_m-aP(t)$, whence

$$P(t)=\frac{Q_m-Q(t)}{a}=P^{(3)}(t). \quad (13)$$

The boundaries of region 3 by the value of $Q(t)$ are the points $Q^{(1)}(t)$ and $Q^{(2)}(t)$:

$$Q^{(1)}(t)\le Q(t)\le Q^{(2)}(t).$$

As can be seen from (13), in this region $P^{(3)}(t)$ decreases linearly with increasing $Q(t)$ from the value $P^{(3)}(t)\Big|_{Q(t)=Q^{(1)}(t)}=\frac{Q_m+RP(t-1)}{a+R}=P^{(1)}_{\max}(t)$ to the value of

$$P^{(3)}(t)\Big|_{Q(t)=Q^{(2)}(t)} = \frac{Q_m + RP(t-1)}{2a+R} = P^{(2)}(t).$$

3. Conditionally maximum profit. Optimal price of goods and maximum profit.

We now find the optimal price of goods and the optimal level of supply of goods to market, providing maximum profit for the seller, if $P(t-1)$, $P^{(1)}_{\max}(t)$ and $P^{(2)}$ satisfy the constraint (6) on $P(t)$. We will solve this problem by zones (zone 1 – shortage of goods, zone 2 – overstocking of the market, zone 3 – the balance of supply and demand, ie dynamic market equilibrium).

1) In the zone 1: $0 < Q(t) < Q^{(1)}(t)$. After substituting $P(t) = P^{(1)}(t)$ in expression (7) for $J(t)$ we have $J(t) = \frac{Q(t)^2}{2R} + \left(P(t-1) - P_1\right)Q(t) + Q^O(t)\left(P_1 - P_2\right) = J^{(1)}(t)$.

As we can see, $J^{(1)}(t)$ increases monotonically with increasing $Q(t)$ according to the linear-quadratic law, reaching the maximum value at the boundary of the region at $Q(t) = Q^{(1)}(t)$: $J^{(1)}_{\max}(t) = J^{(1)}(t)\Big|_{Q(t)=Q^{(1)}(t)}$.

If the balance of goods $Q^O(t)$ from the sale of the previous discrete time interval does not exceed the value of $Q^{(1)}(t)$, then the additional order and delivery of goods to market in the amount of $Q^Z(t-\tau) = Q^{(1)}(t) - Q^O(t)$ (in particular, $Q^Z(t-\tau) = 0$ at $Q^{(1)}(t) = Q^O(t)$) provides the maximum profit. Otherwise, when $Q^{(1)}(t) < Q^O(t)$ we should look for solutions to the problems of profit optimization in areas 2 or 3.

2) In zone 2: $Q(t) > Q^{(2)}(t)$. After substitution in $J(t)$ for this zone (expression (10)) $P(t) = P^{(2)}(t)$, which does not depend on $Q(t)$, we have

$$J(t) = \left(Q_m - aP^{(2)}(t)\right)P^{(2)}(t) - Q(t)P_1 + Q^O(t)\left(P_1 - P_2\right) - \frac{R}{2}\left(P^{(2)}(t) - P(t-1)\right)^2 = J^{(2)}(t).$$

As we see, $J^{(2)}(t)$ decreases monotonically with increasing $Q(t)$ according to the linear law, so that it reaches the highest value in this zone at $Q(t) = Q^{(2)}(t)$:

$$J^{(2)}_{\max}(t) = J^{(2)}(t)\Big|_{Q(t)=Q^{(2)}(t)}.$$

If $Q^O(t) < Q^{(2)}(t)$, then the additional order and supply of goods on the market in the amount of $Q^Z(t-\tau) = Q^{(2)}(t) - Q^O(t)$ provides this maximum profit. If $Q^O(t) > Q^{(2)}(t)$, then $Q^Z(t-\tau) = 0$, and only the value of profit is achieved $J^{(2)}(t)\Big|_{Q(t)=Q^O(t)} < J^{(2)}_{\max}(t)$.

3) In zone 3: $Q^{(1)}(t) \le Q(t) \le Q^{(2)}(t)$. After substituting $P(t) = P^{(3)}(t)$ in $J(t)$ for this zone (formula (10)) we have

$$J(t) = Q(t)\frac{Q_m - Q(t)}{a} - Q(t)P_1 + Q^O(t)\left(P_1 - P_2\right) - \frac{R}{2}\left(\frac{Q_m - Q(t)}{a} - P(t-1)\right)^2 = J^{(3)}(t).$$

This is a convex upward linear-quadratic function of the variable $Q(t)$. The maximum $J^{(3)}(t)$ on the value of $Q(t)$ is reached at the point

$$Q(t) = \frac{R\left(Q_m - aP(t-1)\right) + a\left(Q_m - aP_1\right)}{2a+R} = Q^{(3)}(t).$$

It is obvious that $Q^{(1)}(t) < Q^{(3)}(t) < Q^{(2)}(t)$, ie the point $Q^{(3)}(t)$ of the maximum $J^{(3)}(t)$ lies in zone 3. The maximum value of profit in zone 3 (for $Q^O(t) < Q^{(3)}(t)$)

$$J^{(3)}_{\max}(t) = J^{(3)}(t)\Big|_{Q(t)=Q^{(3)}(t)}.$$

This value is the global maximum, potentially possible for $Q^{O}(t) \le Q^{(3)}(t)$. If $Q^{(3)}(t) < Q^{O}(t) \le Q^{(2)}(t)$, then the global maximum value of profit can not be achieved, and only conditionally the maximum value is achieved (with a fixed $Q^{O}(t)$) within zone 3 lying between (3) $J^{(3)}_{\max}(t)$ and $J^{(2)}(t)$.

Obviously, the global maximum profit can be obtained only if the amount of balance of goods not sold in the previous time interval does not exceed the value of the optimal volume of supply of goods to market: $Q^{O}(t) \le Q^{(3)}(t)$. Otherwise, the seller's profit will be less than the maximum possible. Moreover, if the volume of product balances remains in zone 3, ie lies in the interval $Q^{(3)}(t) < Q^{O}(t) \le Q^{(2)}(t)$, the market will remain in a state of dynamic equilibrium, demand for goods will remain equal to the supply, which will be the balance of the goods). And only for $Q^{O}(t) > Q^{(2)}(t)$ the supply of goods will move to zone 2, and the overstocking of the market will begin.

Conclusions. Thus, the proposed restrictive dynamic mathematical model of the market of one product qualitatively correctly describes the behavior of the market in the conditions of delayed supply of goods to the market. The optimal strategy for the supply of goods requires anticipation of the price of goods and consumer demand ahead of time. This can be done by simulating market behavior.

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